This week

Today:
- Ising model (Binary MRF)
- Inference

Tomorrow
- Inference (INLA)
- Modeling

Lattice data and lattice process

Lattice data: $Y(s_i)$: Data for discrete spatial features (districts, pixels, voxels, non-overlapping catchments)

Lattice process: $Z(s : s \in D_s)$ defined on a finite or countable subset D_s of R^d (d=2).

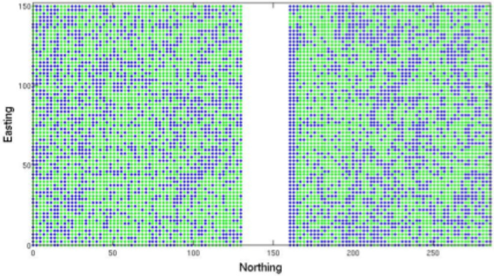
Want to define models 'locally' through full conditional distributions.

Scots Pine Data

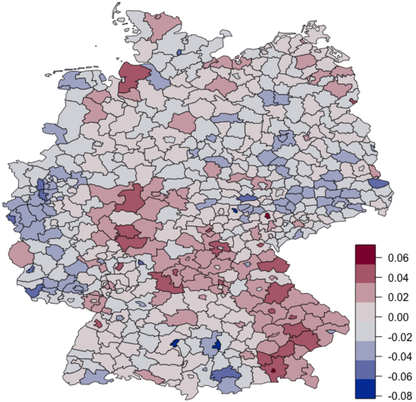
Pedigree 56 unrelated parents, partial diallel design. Original 8160 seedlings.

Spatial location 2.2×2.2m grid, two trail sites.

Data Hight and bad(1) / good(0) branch angle of 4970 26-years-old scots pine.



Sample from GMRF model



Joint distribution from all full conditional distributions

Brook's Lemma

$$\frac{\pi(y)}{\pi(w)} = \prod_{i=1}^{n} \frac{\pi(y_i|y_1, \ldots y_{i-1}, w_{i+1}, \ldots, w_n)}{\pi(w_i|y_1, \ldots y_{i-1}, w_{i+1}, \ldots, w_n)}$$

Set $w = (0, \ldots, 0)$ (or another valid value).

$$\pi(y) = c \cdot \prod_{i=1}^{n} \frac{\pi(y_i|y_1, \ldots y_{i-1}, 0, \ldots, 0)}{\pi(0|y_1, \ldots y_{i-1}, 0, \ldots, 0)}$$

For discrete y:

$$c = \sum_{\text{all combiations of } ys} \prod_{i=1}^{n} \frac{\cdots}{\cdots}$$

For continuous y:

$$c = \int \ldots \int \prod_{i=1}^{n} \frac{\cdots}{\cdots} dy_1 \ldots dy_n$$

Multivariate Normal distribution

Multivariate Normal(MVN) density

$Y = (Y_1, Y_2, \ldots, Y_n)$ is MVN with expected value μ and covarance Σ, $Y \sim MVN(\mu, \Sigma)$ if

$$f(y) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp(-\frac{1}{2}(y-\mu)^T \Sigma^{-1}(t-\mu))$$

Conditional MVN

Let $Y = (Y_1, Y_2)^T$, $\mu = (\mu_1, \mu_2)^T$ and

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

Then the $[Y_1 | Y_2 = a] \sim MVN(\mu_{1|2}, \Sigma_{1|2})$ with

- $\mu_{1|2} = \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(a - \mu_2)$ and
- $\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

MVN: Cannonical parametrization

MVN density, with precision matrix Q

$Y = (Y_1, Y_2, \dots, Y_n)$ is MVN with expected value μ and precision matrix Q, if

$$f(y) = \frac{1}{(2\pi)^{n/2}|Q|^{-1/2}} \exp(-\frac{1}{2}(y-\mu)^T Q (y-\mu))$$

- $Q_{ij} = 0 \Rightarrow y_i$ and y_j NOT neighbors

Cannonical parametrization

If,

$$f(y) \propto exp -\frac{1}{2} y^T Q y - b^T y$$

then y is MVN with $\Sigma = Q^{-1}$ and $\mu = Q^{-1} b$

- $Y_A | Y_B \sim N(\mu_{A|B}, Q_{AA}^{-1})$ with $y_{A|B1} = \mu_A - Q_{AA}^{-1} Q_{AB}(y_B - \mu_B)$

Links

- Example Gibbs sampler Ising model: Link to example Ising model
- Paper composite likelihood : Varin, Reid & Firth, 2011
- Book GMRF (section 2.1-2.2.4) Rue & Held, 2005