Examples of lattice processes and lattice data

- Torstein (Lithology)
- Thea (only catchments)
- Doctor-prescription in France
- Oral cavity cancer in males in Germany, 1986–1990, for 544 districts.
- Scots pine in Sweden
- Temperature change

Doctor-prescription in France

From Cressie and Wikle

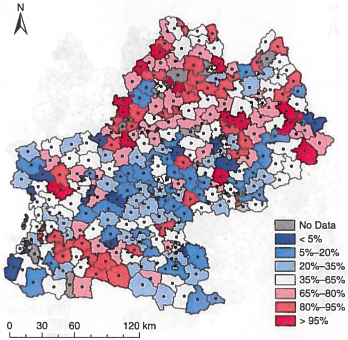

Figure 4.11 Choropleth map showing doctor-prescription amounts per consultation in the cantons of the Midi-Pyrénées (France). The "star" denotes the canton containing Toulouse. Percentiles used

Lattice data and lattice process

Lattice data: $Y(s_i)$: Data for discrete spatial features (districts, pixels, voxels, non-overlapping catchments)

Lattice process: $Z(s : s \in D_s)$ defined on a finite or countable subset D_s of R^d (d=2).

Want to define models 'locally' through full conditional distributions.

Neighborhood and Markov Random Field

Neighborhood: The $N(i)$ *neighborhood* of a random variable Y_i is such that

$$[Y_i | Y_{-i}] = [Y_i | Y(N(i))]$$

Markov Random Field (MRF): A model defined through full conditional distributions.

Q1: Do we know joint distribution from all full conditional distributions?

Brook's Lemma

$$\frac{\pi(y)}{\pi(w)} = \prod_{i=1}^{n} \frac{\pi(y_i | y_1, \ldots y_{i-1}, w_{i+1}, \ldots, w_n)}{\pi(w_i | y_1, \ldots y_{i-1}, w_{i+1}, \ldots, w_n)}$$

Yes, because of Brook's Lemma.

CAR and GMRF

Want model of form

All full conditional Gaussian with:

$$E(Y_i|N(i)) = \sum_{j \in N(i)} c_{ij} y_j$$

$$Var(Y_i|Y(N(i))) = \tau_i^2$$

Book GMRF: Rue & Held, 2005
GMRF = Gaussian Markov Random Fields

Multivariate Normal distribution

Multivariate Normal(MVN) density

$Y = (Y_1, Y_2, \ldots, Y_n)$ is MVN with expected value μ and covarance Σ, $Y \sim MVN(\mu, \Sigma)$ if

$$f(y) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp(-\frac{1}{2}(y-\mu)^T \Sigma^{-1}(t-\mu))$$

Conditional MVN

Let $Y = (Y_1, Y_2)^T$, $\mu = (\mu_1, \mu_2)^T$ and

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

Then the $[Y_1 | Y_2 = a] \sim MVN(\mu_{1|2}, \Sigma_{1|2})$ with

- $\mu_{1|2} = \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(a - \mu_2)$ and
- $\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

Q2: What is needed for a set of full conditional to give valid joint distributions?

Sub-questions:

- Can we find Markov properties in a joint pdf? Yes, Hammersley-Clifford Theorem.

Link G-functions and full conditionals

$$y(\mathbf{s}_i)G_i(y(\mathbf{s}_i)) = \log\left[\frac{\Pr(y(\mathbf{s}_i) \mid \{0(\mathbf{s}_j) : j \neq i\})}{\Pr(0(\mathbf{s}_i) \mid \{0(\mathbf{s}_j) : j \neq i\})}\right], \tag{4.108}$$

where $0(\mathbf{s}_i)$ is shorthand to denote evaluation at $y(\mathbf{s}_i) = 0$; and

$$\begin{aligned} y(\mathbf{s}_i)y(\mathbf{s}_j)G_{ij}(y(\mathbf{s}_i), y(\mathbf{s}_j)) &= \log\left[\frac{\Pr(y(\mathbf{s}_i) \mid y(\mathbf{s}_j), \{0(\mathbf{s}_k) : k \neq i, j\})}{\Pr(0(\mathbf{s}_i) \mid y(\mathbf{s}_j), \{0(\mathbf{s}_k) : k \neq i, j\})}\right] \\ &\quad - \log\left[\frac{\Pr(y(\mathbf{s}_i) \mid \{0(\mathbf{s}_k) : k \neq i\})}{\Pr(0(\mathbf{s}_i) \mid \{0(\mathbf{s}_k) : k \neq i\})}\right]. \end{aligned} \tag{4.109}$$

$$\begin{aligned} & y(\mathbf{s}_i)\ldots y(\mathbf{s}_{j(m)})G_{i,\ldots,j(m)}(y(\mathbf{s}_i), \ldots, y(\mathbf{s}_{j(m)})) \\ & \equiv \sum_{t=0}^{m-2} \sum_{\mathbf{j}_{m-t}^{(-i)} \in T_m^{(-i)}(m-t)} \\ & (-1)^{t-1} \log\left[\frac{\Pr(y(\mathbf{s}_i)|\{y(\mathbf{s}_h) : h \in \mathbf{j}_{m-t}^{(-i)}\}, \{0(\mathbf{s}_k) : k \notin \mathbf{j}_{m-t}\})}{\Pr(0(\mathbf{s}_i)|\{y(\mathbf{s}_h) : h \in \mathbf{j}_{m-t}^{(-i)}, \{0(\mathbf{s}_k) : k \notin \mathbf{j}_{m-t}\})}\right] \\ & + (-1)^{m-1} \log\left[\frac{\Pr(y(\mathbf{s}_i)|\{0(\mathbf{s}_k) : k \neq i\})}{\Pr(0(\mathbf{s}_i)|\{0(\mathbf{s}_k) : k \neq i\})}\right], \end{aligned} \tag{4.117}$$

Ising model

Link to example Ising model

Scots Pine Data

Pedigree 56 unrelated parents, partial diallel design. Original 8160 seedlings.

Spatial location 2.2×2.2m grid, two trail sites.

Data Hight and bad(1) / good(0) branch angle of 4970 26-years-old scots pine.